

Evaluation of Attitude and Orbit Estimation Using Actual Earth Magnetic Field Data

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A single, augmented extended Kalman filter (EKF), which simultaneously and autonomously estimates spacecraft attitude and orbit, has been developed and successfully tested with real magnetometer and gyro data only. Because the Earth magnetic field is a function of time and position, and because time is known quite precisely, the differences between the computed and measured magnetic field components, as measured by the magnetometers throughout the entire spacecraft orbit, are a function of both orbit and attitude errors. Thus, conceivably these differences could be used to estimate both orbit and attitude; an observability study validated this assumption. The results of testing the EKF with actual magnetometer and gyro data, from four satellites supported by the NASA Goddard Space Flight Center Guidance, Navigation, and Control Center, are presented and evaluated. They confirm the assumption that a single EKF can estimate both attitude and orbit when using gyros and magnetometers only.

I. Introduction

MANY future low-budget missions, such as the NASA small and midsize Explorer series and university class explorers, are looking for inexpensive and autonomous approaches to orbit and attitude estimation. One sensor being considered as a prime is the magnetometer because it is reliable and low cost. For these reasons, magnetometers have been the focus of several recent studies. Emphasis has been placed on using only the magnetometer to estimate separately the spacecraft trajectory^{1–3} and attitude.^{4,5} Studies have also been performed that show remarkable accuracy of the magnetometer in estimating attitude when accurate rate information is available.⁶

In this work, we present the design and test results of a single extended Kalman filter (EKF) and its application to actual data obtained from four satellites, namely, Compton Gamma Ray Observatory [(CGRO), North American Aerospace Defense (NORAD) number 20580], Rossi X-Ray Timing Explorer [(RXTE), NORAD number 23757], the Earth Radiation Budget Satellite [(ERBS), NORAD number 15354], and the Total Ozone Mapping Spectrometer-Explorer Platform [(TOMS-EP), NORAD number 23940]. The present work is an extension of the research reported on in Ref. 7, where both attitude and trajectory have been successfully estimated using simulated magnetometer and rate data. In that work large initial errors were applied with a resulting accuracy of 4-km rss in position and less than 1-deg rss in attitude.

In the work presented here, we summarize the derivation of the EKF, highlighting the development of the measurement matrix, which constitutes the crucial element in combining the dependence of the magnetic field residuals on both attitude and trajectory estimation errors. The satellites considered here vary in inclination (from 23 deg for CGRO to 97 deg for TOMS-EP) and in size of the magnetometer quantization error (0.3–6.4 mG). Comparisons are made with operational estimates of position, velocity, and attitude that were very accurate with respect to the filter generated estimation errors, for example, CGRO position measurement error

was less than 1000 m and that of ERBS was within 200 m, etc. These reference values were computed by NASA Goddard Space Flight Center (GSFC) flight dynamics personnel. We show the ability of the filter to overcome large initial errors and a comparison of the final accuracy achieved with each satellite. This work could prove valuable as a prime trajectory and attitude estimation system for satellites with coarse accuracy requirements or as a backup to a prime system in satellites with greater accuracy needs.

II. EKF Algorithm

For clarity of the developments that will follow, we review the special features of the EKF algorithm that was used in this work. The EKF algorithm is based on the following assumed models.

System model:

$$\dot{X}(t) = f[X(t), t] + w(t) \quad (1a)$$

Measurement model:

$$y_{k+1} = h_{k+1}[X(t_{k+1})] + v_{k+1} \quad (1b)$$

where $w(t)$ is a zero mean white process noise, v_{k+1} is a zero mean white sequence measurement error, and $X(t)$ is the state vector. In this work, $X(t)$ is defined as

$$X^T = [a, e, i, \Omega, \omega, \theta, C_d, b^T, q^T] \quad (2)$$

The first six elements of $X(t)$ are the classical Keplerian orbital elements,^{8,9} which determine the spacecraft orbit, position, and velocity, namely, the semimajor axis a , eccentricity e , inclination i , right ascension of the ascending node Ω , argument of perigee ω , and true anomaly θ . C_d is the drag coefficient, b is a vector of the onboard gyro constant drift rates, and q represents the attitude quaternion.

The EKF used in this work is a unified filter that consists of an orbital part and an attitude part. Thus, Eqs. (1) are written as

$$\begin{bmatrix} \dot{X}_o \\ \dot{X}_a \end{bmatrix} = \begin{bmatrix} f_o[X_o(t), t] \\ f_a[X_a(t), t] \end{bmatrix} + \begin{bmatrix} w_o \\ w_a \end{bmatrix} \quad (3a)$$

$$y_{k+1} = h_{k+1}[X_o(t_{k+1}), X_a(t_{k+1})] + v_{k+1} \quad (3b)$$

where

$$X_o^T = [a, e, i, \Omega, \omega, \theta, C_d] \quad (4a)$$

$$X_a^T = [b^T, q^T] \quad (4b)$$

The processes, which are applied to derive the linearized dynamics and measurement models used in the EKF, treat the orbital and the

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attitude parts differently according to the specific physical nature of each part. The difference in treatment is expressed in the development of the measurement model, as well as in the development of the dynamics model. The development of the orbital part of the measurement model is done by the customary partial differentiation, $\partial \mathbf{h} / \partial \mathbf{X}_o$, which yields a Jacobian matrix,¹⁰ whereas, as will be explained later, the attitude part is done using the perturbation method. Similarly, in the development of the orbital part of the dynamics matrix, the customary partial differentiation yields the Jacobian matrix $\partial \mathbf{f}_o / \partial \mathbf{X}_o$. However, as will be shown later, the development of the dynamics model that describes the evolution of the attitude error between measurement updates is entirely different.

The state vector usually consists of three kinds of states: those that describe the dynamics of the system, those that are augmented due to the nonwhite nature of the stochastic processes that drive the dynamics model, and those that are augmented due to the same in the measurement noise. In our case the states a, e, i, Ω, ω , and θ that appear in $\mathbf{X}$ of Eq. (2) are states that describe the system dynamics. The state C_d is a nonwhite state driving the dynamics model, which was augmented via the state augmentation procedure. Similarly, $\mathbf{q}$ is a state vector that describes the dynamics, whereas $\mathbf{b}$, the gyro bias state vector that drives $\mathbf{q}$, is nonwhite and as such is also augmented into the state vector. Sometimes it is also necessary to describe the measurement error as a combination of white noise and colored noise,¹ which, as mentioned before, results in further augmentation of the nonwhite vector into the state vector. However in our filter this was not necessary.

We now proceed to describe the measurement update stage and the propagation stage of the EKF algorithm used in this work.

A. EKF Measurement Update

The measurement update of the state estimate and of the estimation error covariance are performed, respectively, as

$$\hat{\mathbf{X}}_{k+1}(+) = \hat{\mathbf{X}}_{k+1}(-) + K_{k+1} \{ \mathbf{y}_{k+1} - \mathbf{h}_{k+1}[\hat{\mathbf{X}}_{k+1}(-)] \} \quad (5a)$$

$$P_{k+1}(+) = [I - K_{k+1}H_{k+1}]P_{k+1}(-)[I - K_{k+1}H_{k+1}]^T + K_{k+1}R_{k+1}K_{k+1}^T \quad (5b)$$

where $(-)$ is the a priori value, $(+)$ is the a posteriori value, and K_{k+1} is the Kalman gain computed according to

$$K_{k+1} = P_{k+1}(-)H_{k+1}^T [H_{k+1}P_{k+1}(-)H_{k+1}^T + R_{k+1}]^{-1} \quad (6)$$

where

$$H_{k+1} = [H_o \mid H_a]_{k+1} \quad (7)$$

and where H_o is the orbit part of the measurement matrix and H_a is the attitude part. The matrix P_{k+1} is the estimation error covariance matrix, and R_{k+1} is the covariance matrix of the zero mean white sequence $\mathbf{v}_{k+1}$.

Let us denote $\mathbf{y}_{k+1} - \mathbf{h}_{k+1}[\hat{\mathbf{X}}_{k+1}(-)]$ of Eq. (5a) by $\mathbf{z}_{k+1}$. The latter are the data that are used by the EKF to update the a priori state estimate. These data, called effective measurement, are the difference between the measurement and the estimate of the measurement. Therefore, in our case the data, $\mathbf{z}_{k+1}$, used by the filter are computed as

$$\mathbf{z}_{k+1} = \mathbf{B}_{m,k+1} - \mathbf{B}[\hat{\mathbf{X}}_{k+1}(-), t_{k+1}] \quad (8)$$

where $\mathbf{B}_{m,k+1}$ is the magnetic field vector measured by the magnetometer and $\mathbf{B}[\hat{\mathbf{X}}_{k+1}(-), t_{k+1}]$ is the estimated magnetic field vector as a function of the estimated state at time t_{k+1} computed using a 10th-order International Geomagnetic Reference Field (IGRF) model.

Define $\mathbf{x}_{k+1}$ as the difference between the true state vector $\mathbf{X}_{k+1}$ and its estimate, that is,

$$\mathbf{x}_{k+1}^T = [\delta a, \delta e, \delta i, \delta \Omega, \delta \omega, \delta \theta, \delta C_d, \delta \mathbf{b}^T, \boldsymbol{\alpha}^T]_{k+1} \quad (9)$$

A comparison between $\mathbf{X}$, defined in Eq. (2) and $\mathbf{x}$, defined in Eq. (9) reveals that other than $\mathbf{q}$, the latter is a perturbation of the former. In

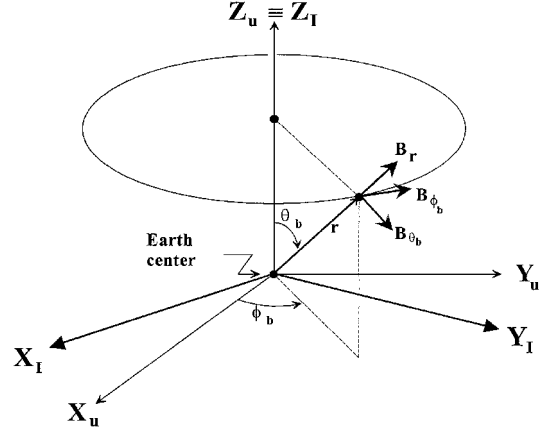


Fig. 1 Definition of the magnetic spherical coordinates.

$\mathbf{x}$ the expression for $\mathbf{q}$ is replaced by $\boldsymbol{\alpha}$. This is done because, as will be shown later, in the EKF update stage we express the attitude error by a vector of small angles, $\boldsymbol{\alpha}$, whereas in the propagation stage we express the attitude by the quaternion $\mathbf{q}$. We use $\mathbf{z}_{k+1}$ [Eq. (8)] to update the estimate, $\hat{\mathbf{x}}_{k+1}$ of $\mathbf{x}_{k+1}$ as follows:

$$\hat{\mathbf{x}}_{k+1} = K_{k+1} \mathbf{z}_{k+1} \quad (10)$$

The updated vector $\hat{\mathbf{x}}_{k+1}$ is used to update the a priori state estimate $\hat{\mathbf{X}}_{k+1}(-)$ as follows:

$$\hat{\mathbf{X}}_{k+1}(+) = \mathbf{g}[\hat{\mathbf{X}}_{k+1}(-), \hat{\mathbf{x}}_{k+1}] \quad (11)$$

The nature of $\mathbf{g}$ will be explained shortly.

1. Development of the Measurement Matrix

Figure 1 shows the geometry of the magnetic field vector. The U coordinates are Earth-fixed coordinates, and the I are inertial coordinates. Both coordinate systems are fixed to the Earth center. The magnetic field vector is expressed in the magnetic spherical coordinates F as follows: $\mathbf{B}_F^T = [B_r, B_{\theta B}, B_{\phi B}]$. Begin the development of the measurement matrix using the perturbation technique, by writing the estimated magnetic field at the spacecraft location as

$$\mathbf{B}(\hat{\mathbf{X}}, t) = \hat{D}_b^I \hat{D}_I^F \hat{\mathbf{B}}_F + \mathbf{v}' \quad (12)$$

where $\hat{\mathbf{B}}_F$ is computed using the IGRF magnetic field model, the estimated position, and time; $\hat{D}_I^F$ is the estimated transformation matrix that transforms from magnetic spherical to inertial coordinates; $\hat{D}_b^I$ is the estimated attitude matrix that transforms the estimated magnetic field vector from the inertial to the body coordinate system; and $\mathbf{v}'$ is the magnetic field model error. (Note that we consider the situation at time point t_{k+1} ; however, we drop the subscript $k+1$ for clarity.) On the other hand, the measured magnetic field vector, as measured by the magnetometer, can be written as

$$\mathbf{B}_m = D_b^I D_I^F \mathbf{B}_F + \mathbf{v}_m \quad (13)$$

where $\mathbf{B}_F$ is the correct magnetic field vector in magnetic spherical coordinates, D_I^F is the correct transformation from the magnetic spherical to the inertial coordinates, D_b^I is the true transformation from inertial to body coordinates, and $\mathbf{v}_m$ is the magnetometer measurement error.

The effective measurement $\mathbf{z}$ is defined as follows:

$$\mathbf{z} = \mathbf{B}_m - \mathbf{B}[\hat{\mathbf{X}}(-), t] = D_b^I D_I^F \mathbf{B}_F + \mathbf{v}_m - \hat{D}_b^I \hat{D}_I^F \hat{\mathbf{B}}_F - \mathbf{v}' \quad (14)$$

Write the transformation of $\hat{\mathbf{B}}_F$ to the estimated body coordinates as

$$\hat{D}_b^I \hat{D}_I^F \hat{\mathbf{B}}_F = D_b^I D_I^F \mathbf{B}_F + \Delta(D_b^I D_I^F \mathbf{B}_F) \quad (15)$$

where $\Delta(D_b^I D_I^F \mathbf{B}_F)$ is the error in the estimation of the magnetic field vector resolved in the body axes. Define

$$\mathbf{v} = \mathbf{v}_m - \mathbf{v}' \quad (16)$$

When the last two equations are used, Eq. (15) becomes

$$\mathbf{z} = -\Delta(D_b^I D_I^F \mathbf{B}_F) + \mathbf{v} \quad (17)$$

Now

$$\Delta(D_b^I D_I^F \mathbf{B}_F) = \Delta D_b^I (D_I^F \mathbf{B}_F) + D_b^I \Delta(D_I^F \mathbf{B}_F) \quad (18)$$

The second term on the right-hand side of Eq. (18) is the part of the effective measurement caused by errors in the orbital states. This term can be developed into $H_o \mathbf{x}_o$, where H_o is the measurement matrix for the orbital states and $\mathbf{x}_o$ are the orbital error states that are the first seven elements of $\mathbf{x}$, which are defined in Eq. (9). The derivation of $H_o \mathbf{x}_o$ is given in Ref. 10 where H_o was developed using partial differentiation,

$$H_o = \left. \frac{\partial \mathbf{h}(\mathbf{X}_o, \mathbf{X}_a)}{\partial \mathbf{X}_o} \right|_{\mathbf{x}=\hat{\mathbf{x}}} \quad (19)$$

The expansion of the first term leads to the measurement matrix for the attitude states. Rewrite that term as

$$\Delta D_b^I (D_I^F \mathbf{B}_F) = \Delta D_b^I \mathbf{B}_I \quad (20)$$

where $\mathbf{B}_I$ is the computed magnetic field vector in inertial coordinates. We define ΔD_b^I , the error in the transformation, as the difference between the correct body coordinates and the computed (estimated) body coordinate system. The computed coordinates are determined by the computed (estimated) transformation matrix $\hat{D}_b^I$, which is the transformation from the inertial to the computed (rather than the correct) body coordinate system. We write it as

$$\hat{D}_b^I = D_c^I = D_c^b D_b^I \quad (21)$$

so that

$$\Delta D_b^I = \hat{D}_b^I - D_b^I = D_c^b D_b^I - D_b^I \quad (22)$$

For small attitude errors, we can assume that the matrix D_c^b is composed of small angles; thus,

$$D_c^b = I - \begin{bmatrix} 0 & -\psi & \vartheta \\ \psi & 0 & -\varphi \\ -\vartheta & \varphi & 0 \end{bmatrix} = I - [\alpha \times] \quad (23)$$

where $[\alpha \times]$ is the cross product matrix of α defined in the last equation and $\alpha^T = [\varphi, \vartheta, \psi]$. The vector α is a vector of three small Euler angles describing the attitude difference between the true and estimated attitudes. These errors are defined about the body x , y , and z axes, respectively. Using Eq. (23) in Eq. (22) yields

$$\Delta D_b^I = (I - [\alpha \times]) D_b^I - D_b^I = -[\alpha \times] D_b^I \quad (24)$$

Substituting Eq. (24) into the first term on the right-hand side of Eq. (18) yields

$$\Delta D_b^I (D_I^F \mathbf{B}_F) = -[\alpha \times] D_b^I \mathbf{B}_I = -[\alpha \times] \mathbf{B}_b = [\mathbf{B}_b \times] \alpha \quad (25)$$

Following the last equation, we substitute the first term of Eq. (18) by $[\mathbf{B}_b \times] \alpha$ and, following the earlier discussion, the second term by $H_o \mathbf{x}_o$. Substituting the result into Eq. (17) yields

$$\mathbf{z} = [-\mathbf{B}_b \times] \alpha + H_o \mathbf{x}_o + \mathbf{v} \quad (26)$$

The error state $\mathbf{x}$ is composed of the orbital error states denoted by $\mathbf{x}_o$, the error in estimating the gyro bias $\delta \mathbf{b}$, and the vector of small angular errors in the attitude α [see Eq. (9)]. Therefore, to relate $\mathbf{z}$ to $\mathbf{x}$ we need to introduce a matrix that relates $\delta \mathbf{b}$ to $\mathbf{z}$. However, $\delta \mathbf{b}$ has no direct influence on $\mathbf{z}$; therefore, we can write Eq. (26) as

$$\mathbf{z} = [H_o | 0_{3 \times 3} | [-\mathbf{B}_b \times]] \mathbf{x} + \mathbf{v} \quad (27)$$

where $0_{3 \times 3}$ is a 3×3 zero matrix. (Note that $\mathbf{b}$ is introduced into the state vector $\mathbf{X}$ because it is a nonwhite vector driving the attitude. To comply with the requirement of the KF that the driving force be

white, $\mathbf{b}$ is augmented with the original state vector. This is a well-known standard procedure. The existence of $\mathbf{b}$ in $\mathbf{X}$ results in the existence of $\delta \mathbf{b}$ in $\mathbf{x}$.) Following Eq. (27), the combined measurement matrix is given as

$$H = [H_o | 0_{3 \times 3} | [-\mathbf{B}_b \times]] = [H_o | H_a] \quad (28)$$

where $H_a = [0_{3 \times 3} | [-\mathbf{B}_b \times]]$. Because $\mathbf{B}_b$ is not known, the magnetic field vector measured by the magnetometer, $\mathbf{B}_m$, is used instead; thus,

$$H_a = [0_{3 \times 3} | [-\mathbf{B}_m \times]] \quad (29)$$

An explicit expression for H_o is given in Ref. 10 using partial differentiation.

2. Measurement Update Methodology

We realize that a new state vector α has emerged in the development of the measurement matrix. This vector of angular errors expresses the attitude error between the true and estimated attitude. We will show shortly that this vector will easily blend with the orbital error states. When Eqs. (26) and (27) are used, the effective measurement at time t_{k+1} is written as

$$\mathbf{z}_{k+1} = H_{k+1} \mathbf{x}_{k+1} + \mathbf{v}_{k+1} \quad (30)$$

where

$$\mathbf{x}_{k+1}^T = [\delta a, \delta e, \delta i, \delta \Omega, \delta \omega, \delta \theta, \delta C_d, \delta \mathbf{b}^T, \alpha^T]_{k+1} \quad (31)$$

Define

$$\mathbf{x}_{o,k+1}^T = [\delta a, \delta e, \delta i, \delta \Omega, \delta \omega, \delta \theta, \delta C_d]_{k+1} \quad (32)$$

From Eqs. (31) and (32), it is obvious that

$$\mathbf{x}_{k+1}^T = [\mathbf{x}_{o,k+1}^T, \delta \mathbf{b}^T, \alpha^T]_{k+1} \quad (33)$$

Similar to $\mathbf{x}_{o,k+1}^T$, define

$$\mathbf{X}_{o,k+1}^T = [a, e, i, \Omega, \omega, \theta, C_d]_{k+1} \quad (34)$$

then from Eqs. (2) and (4) we obtain

$$\mathbf{X}_{k+1}^T = [\mathbf{X}_{o,k+1}^T, \mathbf{b}^T, \mathbf{q}_{k+1}^T] \quad (35)$$

With Eq. (35) we can now explain the nature of $\mathbf{g}$ in Eq. (11). Indeed, the update of the state vector estimate is a function of both $\hat{\mathbf{X}}_{k+1}(-)$ and $\hat{\mathbf{x}}_{k+1}$, which is computed using Eq. (9). Whereas the orbital and the gyro bias parts of $\hat{\mathbf{X}}_{k+1}(-)$ are updated additively as follows:

$$\begin{bmatrix} \hat{\mathbf{X}}_{o,k+1}(+) \\ \hat{\mathbf{b}}_{k+1}(+) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{X}}_{o,k+1}(-) \\ \hat{\mathbf{b}}_{k+1}(-) \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{x}}_{o,k+1} \\ \hat{\delta \mathbf{b}}_{k+1} \end{bmatrix} \quad (36)$$

the attitude part is updated multiplicatively as described next. It is well known that the three small angles in the vector $\hat{\alpha}$ can be used to construct a quaternion $\hat{\mathbf{q}}$ as follows¹¹:

$$\hat{\mathbf{q}}^T = \left[\frac{1}{2} \hat{\varphi}, \frac{1}{2} \hat{\vartheta}, \frac{1}{2} \hat{\psi}, 1 \right] = \left[\frac{1}{2} \hat{\alpha}^T, 1 \right] \quad (37)$$

where $\hat{\mathbf{q}}$ is the estimate of $\mathbf{q}$ that expresses the small attitude difference between our best estimate of the attitude and the correct attitude. In other words, if one rotates the correct coordinate system, whose orientation is expressed by $\mathbf{q}_{k+1}$, through the rotation expressed by $\hat{\mathbf{q}}$, one reaches the current (a priori) estimate of the orientation expressed by $\hat{\mathbf{q}}_{k+1}(-)$. This is represented by

$$\hat{\mathbf{q}}_{k+1}(-) = \mathbf{q}_{k+1} \otimes \hat{\mathbf{q}}_{k+1} \quad (38)$$

where $\otimes$ denotes a quaternion product. Postmultiplying Eq. (38) by the inverse of $\mathbf{q}_{k+1}$ yields

$$\mathbf{q}_{k+1} = \hat{\mathbf{q}}_{k+1}(-) \otimes \mathbf{q}_{k+1}^{-1} \quad (39a)$$

If $\mathbf{q}_{k+1}$ is known precisely, then the operation $\hat{\mathbf{q}}_{k+1}(-) \otimes \mathbf{q}_{k+1}^{-1}$ yields $\hat{\mathbf{q}}_{k+1}$, but because we have only an estimate $\hat{\mathbf{q}}_{k+1}$ of $\mathbf{q}_{k+1}$,

the product $\hat{\mathbf{q}}_{k+1}(-) \otimes \hat{\mathbf{d}}_{k+1}^{-1}$ yields only the estimate of $\mathbf{q}_{k+1}$. Because $\hat{\mathbf{d}}_{k+1}$ is the a posteriori estimate of $\mathbf{d}_{k+1}$, this new estimate of $\mathbf{q}_{k+1}$ is the a posteriori estimate. In other words,

$$\hat{\mathbf{q}}_{k+1}(+) = \hat{\mathbf{q}}_{k+1}(-) \otimes \hat{\mathbf{d}}_{k+1}^{-1} \quad (39b)$$

The last equation can be viewed as a correction to $\hat{\mathbf{q}}_{k+1}(-)$ that yields $\hat{\mathbf{q}}_{k+1}(+)$. To assure the normality of the quaternion, normalization is performed at this stage, which yields

$$\hat{\mathbf{q}}_{k+1}(+) = \hat{\mathbf{q}}_{k+1}(+) / |\hat{\mathbf{q}}_{k+1}(+)| \quad (39c)$$

Following Eq. (35) the full updated state vector is given by

$$\hat{\mathbf{X}}_{k+1}^T(+) = [\hat{\mathbf{X}}_{o,k+1}^T(+), \hat{\mathbf{b}}_{k+1}^T, \hat{\mathbf{q}}_{k+1}^T(+)] \quad (40)$$

B. EKF Propagation

Recall the dynamics equation (3a)

$$\begin{bmatrix} \dot{\mathbf{X}}_o \\ \dot{\mathbf{X}}_a \end{bmatrix} = \begin{bmatrix} \mathbf{f}_o[\mathbf{X}_o(t), t] \\ \mathbf{f}_a[\mathbf{X}_a(t), t] \end{bmatrix} + \begin{bmatrix} \mathbf{w}_o \\ \mathbf{w}_a \end{bmatrix} \quad (3a)$$

The propagation of the covariance is given as

$$\mathbf{P}_{k+1}(-) = \mathbf{A}_k[\mathbf{X}_k(+)]\mathbf{P}_k(+)\mathbf{A}_k^T[\mathbf{X}_k(+)] + \mathbf{Q}_k \quad (41)$$

where $\mathbf{Q}_k$ is the covariance matrix resulting when discretizing the spectral density matrix of $\mathbf{w}(t)$ and $\mathbf{A}_k$ is the approximated transition matrix. $\mathbf{A}_k$ is computed using the following first-order Taylor series expansion:

$$\mathbf{A}_k = \mathbf{I} + \mathbf{F}_k \cdot \Delta T \quad (42)$$

where ΔT is the time interval between gyro measurements. As described earlier, the dynamics matrix of the orbit part is obtained by computing the Jacobian $\mathbf{F}_{o,k}$ as follows:

$$\mathbf{F}_{o,k} = \left. \frac{\partial \mathbf{f}[\mathbf{X}_o(t), t]}{\partial \mathbf{X}} \right|_{\mathbf{X} = \hat{\mathbf{X}}_k(-)} \quad (43)$$

The actual computation of $\mathbf{F}_{o,k}$ is based on the orbital dynamics equations, given subsequently.

Based on Eq. (1), the propagation of the state estimate is performed by solving the differential equation

$$\dot{\hat{\mathbf{X}}}(t) = \mathbf{f}[\hat{\mathbf{X}}(t), t] \quad (44)$$

The updated estimate of the state vector $\hat{\mathbf{X}}_k(+)$ serves as the initial condition of the solution, and the final solution at time t_{k+1} is $\hat{\mathbf{X}}_{k+1}(-)$. The dynamics equation (44) consists of the orbital dynamics equation, which in our case is nonlinear, and the attitude kinematics, which is linear.

1. Nominal Dynamics

Orbital dynamics. The orbital dynamics describe the motion of a mass point in a central force field including drag.¹² There are several possibilities to describe the orbital dynamics. The most natural possibility is by position, velocity, and acceleration expressed in a Cartesian coordinate system. The advantage of this choice is in its ability to handle orbits that are completely circular, but its disadvantage is in the fast change of the state that describes the dynamics. This in turn requires an increased computational load. One can also choose equinoctial variables¹³ to describe the orbital dynamics. This choice, too, enables the handling of completely circular orbits. Moreover, all but one variable of the equinoctial variables change slowly (see Ref. 9, p. 143). Finally, one can use Keplerian parameters, which, similarly to the equinoctial variables, include only one fast varying parameter. Although these parameters cannot describe perfectly circular orbits, as will be seen later they can handle nearly circular orbits. For this reason and because Keplerian parameters are widely used and well understood, we chose them for the present study.

The differential equations for each of the orbital elements of the state vector $\mathbf{X}$, including the drag coefficient, are given as¹⁰

$$\dot{x}_1 = \dot{a} = \frac{2a}{r}(2a - r)\frac{f_t}{V} \quad (45a)$$

$$\dot{x}_2 = \dot{e} = \frac{2f_t}{V}(\cos \theta + e) + \frac{f_n}{V}\frac{r}{a}\sin \theta \quad (45b)$$

$$\dot{x}_3 = \dot{i} = \frac{rf_h}{h}\cos \theta^* \quad (45c)$$

$$\dot{x}_4 = \dot{\Omega} = \frac{rf_h}{h \cdot \sin i}\sin \theta^* \quad (45d)$$

$$\dot{x}_5 = \dot{\omega} = \frac{1}{e} \left[\frac{2f_t}{V}\sin \theta - \frac{f_n}{V} \left(2e + \frac{r}{a}\cos \theta \right) \right] - \dot{\Omega}\cos i \quad (45e)$$

$$\dot{x}_6 = \dot{\theta} = \frac{h}{r^2} - \dot{\Omega}\cos i - \dot{\omega} \quad (45f)$$

$$\dot{x}_7 = \dot{C}_d = 0 \quad (45g)$$

where f_t , f_n , and f_h are the along track, radial, and cross track perturbing accelerations due to the effect of drag and where

$$\theta^* = \omega + \theta \quad (46a)$$

$$h = \sqrt{\mu_E a(1 - e^2)} \quad (46b)$$

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta} \quad (46c)$$

$$V = \sqrt{2\mu_E \left(\frac{1}{r} - \frac{1}{2a} \right)} \quad (46d)$$

and μ_E is the Earth gravitation constant.

Attitude dynamics. The linear differential equation governing the dynamics of the attitude quaternion is given by

$$\dot{\mathbf{q}} = \frac{1}{2}\tilde{\Omega}\mathbf{q} \quad (47a)$$

where $\tilde{\Omega}$ is a 4×4 skew-symmetric matrix containing the elements of the spacecraft rate vector as measured by the gyros.¹¹ We assume that the gyros have bias, and using the state augmentation approach,¹⁴ mentioned in Sec. II, the three bias states were included in the state vector [Eqs. (2) and (4b)]. The bias states model is

$$\dot{\mathbf{b}} = 0 \quad (47b)$$

Note that although the correct modeling of bias is as shown in Eq. (47b), we added white driving noise for filter stability.¹⁵

2. Error-State Dynamics

Orbital error-state dynamics. The error dynamics model can be written as follows:

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_o & 0 \\ 0 & \mathbf{F}_a \end{bmatrix} \quad (48)$$

where $\mathbf{F}_o$ is the orbital part and $\mathbf{F}_a$ is the attitude part of the error-state dynamics. From Eq. (32) it is obvious that the dimension of $\mathbf{F}_o$ is 7×7 . The elements of this matrix are obtained when applying the partial differentiation of Eq. (43) to Eqs. (45). In the case where the drag coefficient C_d is not being estimated and the effect of J_2 is neglected, the nonzero elements of $\mathbf{F}_o$ are as follows.¹⁰ Let

$$f_6 = \frac{\sqrt{\mu_E}(1 + e \cdot \cos \theta)^2}{[a(1 - e^2)]^{\frac{3}{2}}} \quad (49)$$

then

$$f_{a6,1} = -\frac{3}{2a} f_6(X) \quad (50a)$$

$$f_{a6,2} = \left[\frac{3e}{1-e^2} + \frac{2\cos\theta}{1+e\cdot\cos\theta} \right] f_6(X) \quad (50b)$$

$$f_{a6,6} = -\frac{2e\cdot\sin\theta}{1+e\cdot\cos\theta} f_6(X) \quad (50c)$$

In the more general case where C_d and the influence of J_2 are considered, the development of F_o is more lengthy and is found in Refs. 10 and 12.

Attitude error-state dynamics. As mentioned earlier, the development of F_a , the dynamics model that describes the evolution of the attitude and gyro bias errors between measurement updates, is done entirely differently than that of F_o . From Eq. (37),

$$dq^T = \left[\frac{1}{2}\alpha^T, 1 \right] \quad (51)$$

Generalization of the relation expressed in Eq. (38) yields

$$\dot{q} = q \otimes dq \quad (52)$$

from which we obtain

$$\dot{q} = \dot{q} \otimes dq + q \otimes \dot{q} \quad (53)$$

Noting that $q^{-1} \otimes q = u$, where u is the unity quaternion, we get from the last equation

$$d\dot{q} = q^{-1} \otimes \dot{q} - q^{-1} \otimes \dot{q} \otimes dq \quad (54)$$

We can replace $\dot{q}$ by $\frac{1}{2}q \otimes \tilde{\omega}$, where $\tilde{\omega}$ is ω presented in quaternion form (see Ref. 16), and similarly replace $\dot{q}$ by $\frac{1}{2}\hat{q} \otimes \tilde{\omega}_m$, where $\tilde{\omega}_m$ is the measured angular rate ω_m , presented also in quaternion form. The last equation then becomes

$$d\dot{q} = q^{-1} \otimes \frac{1}{2}\hat{q} \otimes \tilde{\omega}_m - q^{-1} \otimes \frac{1}{2}q \otimes \tilde{\omega} \otimes dq \quad (55)$$

By the use of Eq. (52) as well as the relation $\omega_m = \omega + d\omega$ (where $d\omega$ is the angular rate measurement error, which, in our case, is the gyro drift rate), Eq. (55) can be written as

$$d\dot{q} = q^{-1} \otimes \frac{1}{2}q \otimes dq \otimes (\tilde{\omega} + d\tilde{\omega}) - q^{-1} \otimes \frac{1}{2}q \otimes \tilde{\omega} \otimes dq \quad (56)$$

Equation (56) can be written as

$$d\dot{q} = \frac{1}{2}[dq \otimes \tilde{\omega} - \tilde{\omega} \otimes dq] + \frac{1}{2}dq \otimes d\tilde{\omega} \quad (57)$$

When the rules of quaternion multiplication are used (for example, see Ref. 11, p. 759), the real part of the quaternion $[dq \otimes \tilde{\omega} - \tilde{\omega} \otimes dq]$ is identically zero, that is,

$$[dq \otimes \tilde{\omega} - \tilde{\omega} \otimes dq]_r = 0 \quad (58a)$$

where r is the real part. Similarly, we find that

$$[dq \otimes d\tilde{\omega}]_r = -d\omega \cdot \frac{1}{2}\alpha \approx 0 \quad (58b)$$

where $d\omega$ is a vector. For the imaginary part of the quaternion $[dq \otimes \tilde{\omega} - \tilde{\omega} \otimes dq]$ we find that

$$[dq \otimes \tilde{\omega} - \tilde{\omega} \otimes dq]_{\text{Im}} = 2\omega \times \frac{1}{2}\alpha \quad (59a)$$

where Im is the imaginary part and ω is a vector. For the imaginary part of the quaternion $[dq \otimes d\tilde{\omega}]$ we find that

$$[dq \otimes d\tilde{\omega}]_{\text{Im}} = d\omega + d\omega \times \frac{1}{2}\alpha \approx d\omega \quad (59b)$$

When Eqs. (58) and (59) are used, Eq. (57) becomes

$$\dot{\alpha} = \omega \times \alpha + d\omega \quad (60a)$$

The term $d\omega$ is the error introduced by the gyros, which we assume it consists of two parts: a very low-frequency signal and a wideband

signal. We modeled the former as bias and the latter as white noise; thus,

$$d\omega = b + n \quad (60b)$$

where b is the bias state whose model was given in Eq. (47b). As mentioned earlier, b was augmented with the original state vector [see Eq. (2)]. In the error state vector x defined in Eq. (10), the error in estimating b is denoted δb , where obviously

$$\delta \dot{b} = 0 \quad (60c)$$

The resulting matrix F_a describes the propagation of $\hat{\alpha}$ as well as the estimate of the gyro constant drift rate. It is a 6×6 matrix. In view of Eq. (60), the nonzero elements of F_a are as follows:

$$f_{a4,1} = 1 \quad (61a)$$

$$f_{a4,5} = \omega_z \quad (61b)$$

$$f_{a4,6} = -\omega_y \quad (61c)$$

$$f_{a5,2} = 1 \quad (61d)$$

$$f_{a5,4} = -\omega_z \quad (61e)$$

$$f_{a5,6} = \omega_x \quad (61f)$$

$$f_{a6,3} = 1 \quad (61g)$$

$$f_{a6,4} = \omega_y \quad (61h)$$

$$f_{a6,5} = -\omega_x \quad (61i)$$

III. Filter Testing

The observability of the unified filter was examined first using the following analysis. Let us denote the transition matrix, which corresponds to F of Eq. (48), by $A_k(\hat{X})$. This matrix transforms x_{k-1} , the error in the state estimate at time t_{k-1} , to x_k , the error in the state estimate at time t_k . If at a certain time point t_m the initial error in the state estimate, denoted by x_0 , can be computed, then for our purposes the system is observable. This is so because in the EKF, x is defined as the difference between the system state vector X and its known estimate $\hat{X}$. Adopting the common approach to the proof of complete observability of a discrete linear system, we express the first m measurements as follows:

$$y_j = H_j \left[\prod_{i=0}^j A_i(\hat{X}) \right] x_0 \quad j = 0, 1, \dots, m-1 \quad (62)$$

where $A_0(\hat{X}) = I$. Form the matrix equation

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{m-1} \end{bmatrix} = \begin{bmatrix} H_0 \\ H_1 A_1(\hat{X}) \\ H_2 A_2(\hat{X}) A_1(\hat{X}) \\ \vdots \\ H_{m-1} A_{m-1}(\hat{X}) \cdots A_1(\hat{X}) \end{bmatrix} x_0 \quad (63)$$

If there are n independent rows in the right-hand side matrix (the observability matrix) in Eq. (63), then, of course, x_0 can be computed; hence, the state is observable. When the filter with CGRO data was tested, it was found that after 5 measurements there were 13 independent rows in the observability matrix, and because the size of the state vector was 13, the system was observable.

After finding that the filter operated on an observable CGRO system, the EKF algorithm described in the preceding was tested on data obtained from four satellites. The four satellites were CGRO, RXTE, ERBS, and TOMS. The launch dates of these satellites and the start time of the data used in the testing is presented in Table 1. The satellites varied in altitude, inclination, resolution of the telemetry of the magnetometer data (quantization error), and frequency at

Table 1 Satellite launch times

Satellite	Launch date	Start of data
RXTE	30 Dec. 1995	8 Sept. 1996, 19 h 53 min 11.769 s
ERBS	5 Oct. 1984	25 Dec. 1984, 19 h 22 min 12.000 s
CGRO	5 April 1991	3 May 1993, 14 h 12 min 36.841 s
TOMS	2 July 1996	15 March 1997, 19 h 5 min 14.514 s

Table 2 Satellite information

Satellite	<i>h</i> , km	<i>i</i> , deg	Error,	
			mG	ΔT , s
RXTE	580	23.0	0.3	2
ERBS	614	57	6.4	16
CGRO	340	28.5	0.3	3–4
TOMS	483	97	4	33–34

Table 3 Initial rss errors

Satellite	Position, km	Velocity, km/s	Attitude, deg
RXTE	2617	2.6	15.7
ERBS	1000	1.1	11.4
CGRO	1098	1.1	12.9
TOMS	2389	2.6	14

Table 4 Average rss errors (excluding transient effects)

Satellite	Position, km	Velocity, km/s	Attitude, deg
RXTE ^a	15	0.015	1.0
ERBS	25	0.03	1.4
CGRO	20	0.02	0.2
TOMS	20	0.025	0.75

^aAverages are for the last three orbits.

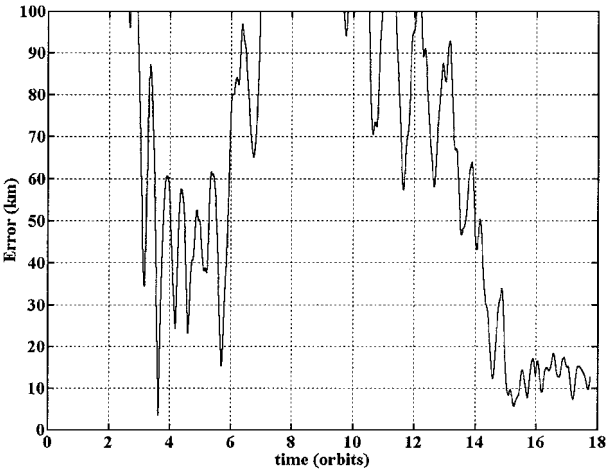


Fig. 2 Evolution of the RXTE rss position estimation error, from initial error of 2617 km.

which the magnetometer telemetry was received. Table 2 summarizes these variations, where *h* is the average altitude (the orbits were nearly circular), *i* is the inclination, error is the quantization error of the telemetry, and ΔT is the time between magnetometer measurements. In addition, the accuracy of the gyroscopes varied. [Note that the gyro data were needed for computing the attitude dynamics model of Eqs. (47a) and (61).] The CGRO and RXTE gyros were considerably more accurate than those of TOMS and ERBS. (CGRO used the Teledyne DRIRU-II gyro, whose drift rate stability was 0.003 deg/h over 6 h. RXTE used the Kearfott SKIRU-DII, which was equivalent to the DRIRU-II gyro. TOMS and ERBS used less accurate gyros, whose drift rate stability was about an order of magnitude larger.) The initial estimate of each satellite position, velocity, and attitude determines the initial orbital parameters and the initial quaternion, which constitute the initial state estimate.

The initial estimation errors were determined by comparing the filter position, velocity, and attitude with estimates of position, velocity, and attitude computed by NASA GSFC flight dynamics personnel. The initial errors, which correspond to the initial estimate chosen for each satellite, are given in Table 3. The errors displayed are the rss errors.

Figures 2–9 show the evolution of the position and attitude estimation errors for each of the four satellites. Figures 2 and 3 show, respectively, the RXTE rss position and attitude errors, Figs. 4 and 5

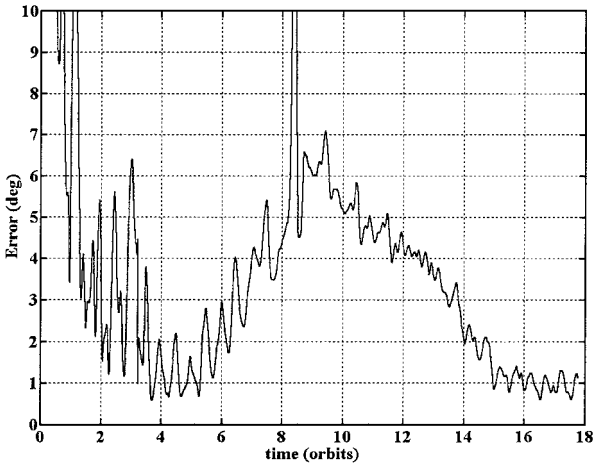


Fig. 3 Evolution of the RXTE rss attitude estimation error, from initial error of 15.7 deg.

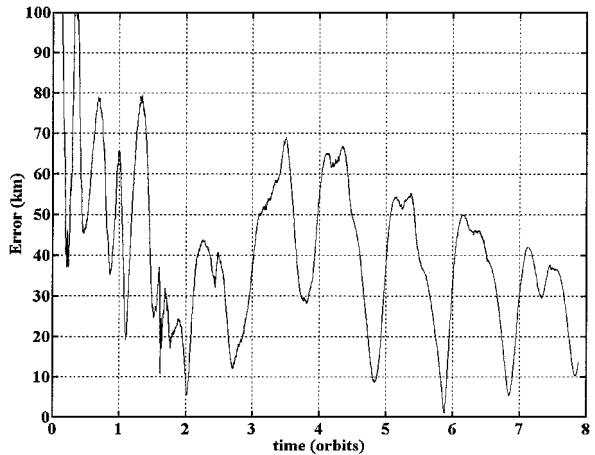


Fig. 4 Evolution of the ERBS rss position estimation error, from initial error of 1000 km.

show ERBS results, Figs. 6 and 7 show the CGRO results, and, finally, Figs. 8 and 9 show TOMS results. The velocity errors are not shown, but for each satellite they have a shape similar to the position error.

Table 4 summarizes the average rss errors for position, velocity, and attitude for each satellite (excluding transients). Note that the RXTE transients last for 15 orbits; therefore, the RXTE rss errors are averaged only over the last three orbits. For ERBS, CGRO, and TOMS, the final position errors are comparable, with averages of 20–30 km. All three also converge quickly, including TOMS, which started with a 2389-km position error. The position estimates from all three appear to still be converging. However, to assure convergence for all four satellites, data of one more Earth revolution inside the orbit were necessary. That is, data of some additional 16 orbits were required. Unfortunately the data spans that were available for these satellites were limited. However, the resemblance of the results to results obtained via simulations where, of course, data were not limited, confirm the convergence of the results with real data.

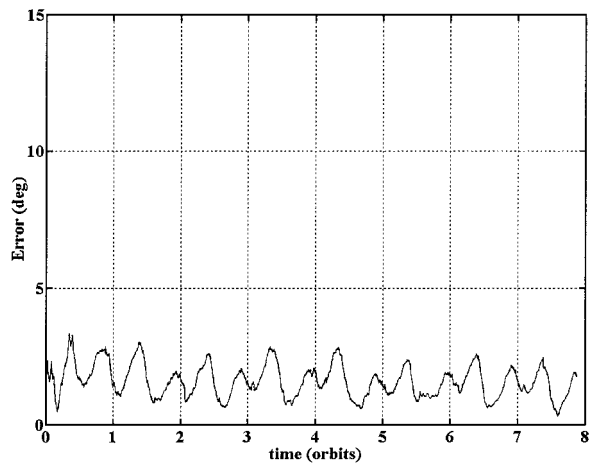


Fig. 5 Evolution of the ERBS rss attitude estimation error, from initial error of 11.4 deg.

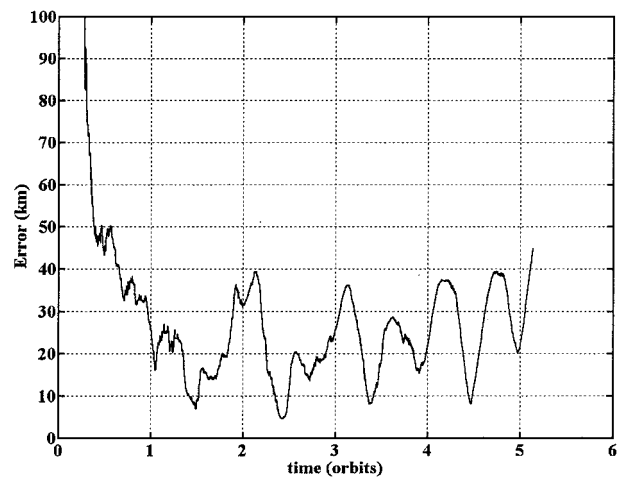


Fig. 8 Evolution of the TOMS rss position estimation error, from initial error of 2389 km.

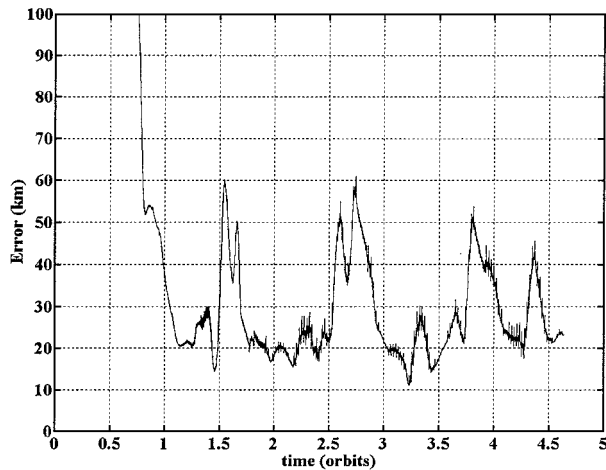


Fig. 6 Evolution of the CGRO rss position estimation error, from initial error of 1098 km.

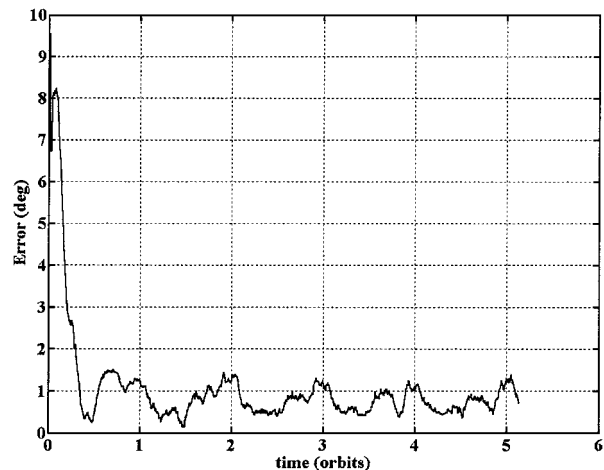


Fig. 9 Evolution of the TOMS rss attitude estimation error, from initial error of 14 deg.

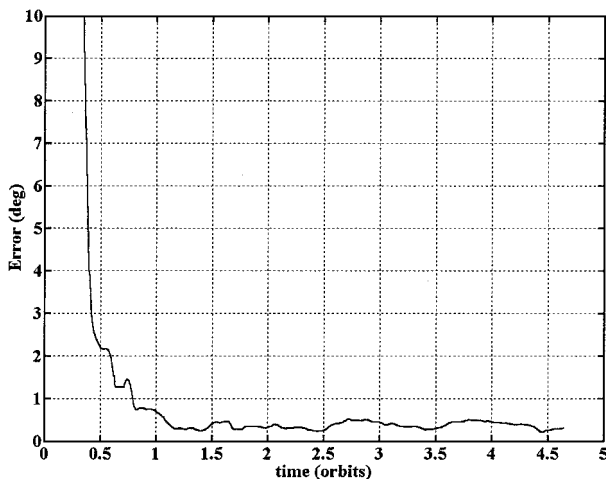


Fig. 7 Evolution of the CGRO rss attitude estimation error, from initial error of 12.9 deg.

CGRO has the lowest attitude error. The final average rss attitude error is 0.2 deg. The CGRO data were extensively calibrated before its use, the magnetometer quantization error was small, and the measurement frequency, particularly of the gyro data, was high (0.256 s between gyro readings). Surprisingly, the TOMS attitude errors were also relatively small, with a final average of 0.75 deg. This was a result of the high inclination for TOMS, which resulted in good observability, despite the large quantization error, the low

measurement frequency, and the uncalibrated magnetometer data. ERBS had the highest attitude error, with a final average of 1.4 deg. This was because ERBS had the largest quantization error and also a low measurement frequency.

RXTE took considerably longer to converge, and both the attitude and position error results show a divergence in the middle of the span. Note that RXTE underwent three attitude maneuvers during this span, which apparently led to a deviation in the drag coefficient, which then led to the increased errors. Once the drag coefficient converged again, the position results improved and for the last three orbits were very good with an average rss position error of approximately 15 km. The final average rss attitude errors were approximately 1 deg. The RXTE data were not calibrated at all and contain unknown and uncompensated disturbances, which contributed to the larger final attitude error.

These results compare favorably with results obtained in Ref. 17, where only magnetometers were used. In particular the present results fit the accuracy predicted in Ref. 17. They also fit results reported in Ref. 18, which were obtained with simulated data and are better than the results presented in Ref. 19.

The gyros that were used onboard these spacecraft were well calibrated; therefore, their bias was negligible. Consequently, the gyro bias states were deactivated; however, to examine the ability of the filter to estimate bias, an artificial bias of 1.0 mrad/s was added to the CGRO gyro output and the filter was rerun with active bias states. The bias states were estimated without corrupting the estimation of the other states. Figure 10 presents the time history of the estimated bias vs its correct value in the three-body axes.

Although successful in estimating the larger part of the gyro bias, the filter was unable to estimate very low gyro bias values, apparently

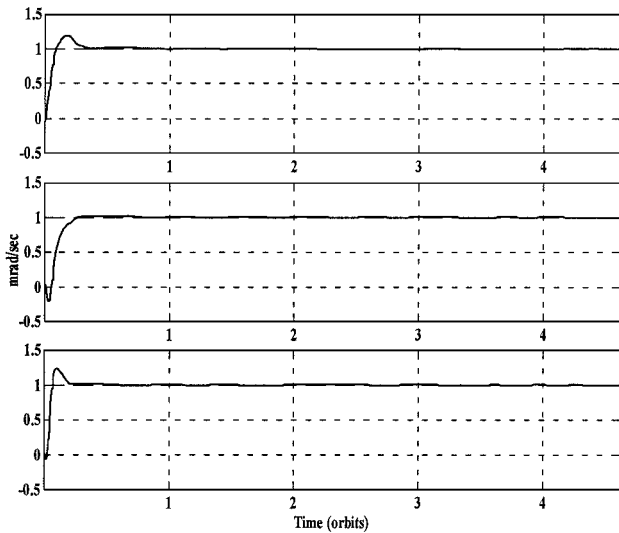


Fig. 10 CGRO artificial bias estimates as a function of time (orbits).

because of the large measurement noise. Moreover, when the bias states were activated and the bias to be estimated was negligible, the orbit and attitude estimates degraded considerably. On the other hand, in the presence of large gyro bias, the inclusion of the bias states in the filter was imperative to obtain good orbit and attitude estimates. A byproduct of this inclusion was the estimation of the major part of the gyro bias. The degradation in performance of the filter, when the bias states of the estimator were activated despite the lack of actual bias, has been explained before by the information dilution theorem.²⁰

IV. Conclusions

This paper presented a new unified EKF algorithm for estimating spacecraft orbit and attitude based on measurements of the magnetic field and the angular velocity of the spacecraft. To test this algorithm, it was first established that the system, which the EKF had to estimate, was indeed observable for at least the CGRO satellite. Next, magnetometer data and gyro data from four satellites were processed by the EKF. All four converged to final averages of 15–30 km in position, 0.015–0.03 km/s in velocity, and 0.2–1.5 deg in attitude (all rss). Additional results indicated that the EKF could converge from extremely large initial position and velocity errors; CGRO and TOMS, for example, overcame initial position errors exceeding 5000 km.

The data from each of the satellites differed in quantization error and measurement frequency, and the satellites were at four different inclinations. The higher inclinations for ERBS and TOMS resulted in quick convergence and for TOMS gave good final results despite inaccuracies in the data and a large time between measurements. The coarse quantization of the ERBS telemetry and the infrequency of the measurements resulted in the lowest accuracy. On the other hand, the extensive calibration of the CGRO data, the small quantization errors, and the high data frequency contributed to the quick convergence and the higher accuracy achieved with CGRO. It was observed that RXTE required more time to converge, with good final results in position. The RXTE magnetometer data contained disturbances that were not calibrated and, hence, contributed to the errors in attitude.

Although the filter was designed to also estimate gyro bias, in our experiments we deactivated the bias states because the available gyro data had been calibrated (which removed the gyro bias). To test the filter performance when gyro bias is present, artificial gyro bias was added to the CGRO data, and the filter bias states were

activated. It was found that the filter estimated the gyro bias without degrading the quality of the attitude and orbit estimation. The filter successfully estimated the larger part of the gyro bias, but was unable to estimate very low gyro bias values, probably because of the large measurement noise. Moreover, when trying to estimate negligible gyro drifts, the orbit and attitude estimates degraded considerably. However, in the presence of large gyro bias, it was necessary to activate the bias states in the filter to obtain good orbit and attitude estimates. A byproduct of this inclusion was the estimation of the major part of the gyro bias.

In summary it was shown that magnetometer and gyro measurements onboard low Earth-orbiting satellites are sufficient for coarse orbit as well as attitude determination when processed by the single EKF developed in this work.

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